Sample Final C, Math 1554

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name ____________________ Last Name ____________________

GTID Number: ______________________

Student GT Email Address: _________________________@gatech.edu

Section Number (e.g. A4, QH3, etc.) _______ TA Name _________________________

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- Any work done on scratch paper will not be collected and will not be graded.

You do not need to justify your reasoning for questions on this page.

1. (6 points) Circle **true** if the statement is true, otherwise, circle **false**.

   (a) A product of invertible matrices is also invertible.

   true **false**

   (b) Regardless what A and $\vec{b}$ are, there is always at least one least-squares solution $\hat{x}$ to $A\vec{x} = \vec{b}$. **Assume A is mxn and b is in R^m so that Ax=b is defined.**

   true **false**

   (c) If $A\vec{x}_0 = \vec{b}$, and $A\vec{y} = \vec{0}$, then $\vec{x} = \vec{x}_0 - 5\vec{y}$ is a solution to $A\vec{x} = \vec{b}$.

   true **false**

   (d) An example of a quadratic form is the polynomial $7x_1^2 + 5x_2^2 - 10x_1x_2 + x_2$.

   true **false**

   (e) If a matrix is invertible then it is also diagonalizable.

   true **false**

   (f) A $n \times n$ matrix A and its echelon form E have the same eigenvalues.

   true **false**

2. (6 points) Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**.

   (a) The columns of matrix A are linearly independent, and NullA^T is not trivial.

   possible **impossible**

   (b) A is $n \times n$, $\lambda \in \mathbb{R}$ is an eigenvalue of A, and $\dim(\text{Col}(A - \lambda I)^\perp) = 0$.

   possible **impossible**

   (c) Stochastic matrix P has zero entries and is regular.

   possible **impossible**

   (d) A is a square matrix that is not diagonalizable, but A^2 is diagonalizable.

   possible **impossible**

   (e) A is 5×4, $A\vec{x} = \vec{b}$ has three free variables, and $\dim(\text{Row}(A)^\perp) = 3$.

   possible **impossible**

   (f) A $m \times n$ matrix A has linear transformation T_A. The map T_A can be one-to-one but not onto.

   possible **impossible**

You do not need to explain your reasoning for questions on this page.

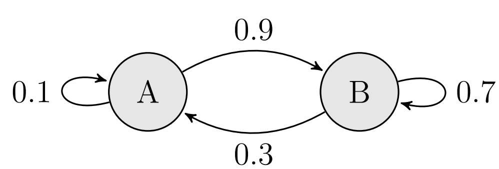
3. (8 points) If possible, give an example of the following. If it is not possible, write "not possible".

   (a) A matrix that is 2×4, in reduced echelon form, with the dimension of column space being 3, and dimension of null space is 1.

   (b) A 3×4 matrix with orthonormal columns.

   (c) A 3×2 matrix A in reduced echelon form so that $A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   (d) A stochastic matrix for the Markov Chain below.



   (e) A 2×2 matrix whose column space is the line $2x_1 + x_2 = 0$, and whose null space is the line $4x_1 - x_2 = 0$.

4. (10 points) A has exactly two distinct eigenvalues, which are -2, and 1.

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

(a) Construct an eigenbasis for eigenvalue $\lambda = -2$.

(b) Construct an eigenbasis for eigenvalue $\lambda = 1$.

(c) If possible, construct matrices P and D such that $A = PDP^T$, and P is a matrix with orthonormal columns and D is a diagonal matrix.

5. (10 points) Construct the singular value decomposition of $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

6. (6 points) Circle **true** if the statement is true, otherwise, circle **false**. You do not need to explain your reasoning.

   (a) For any $n \times n$ matrix A, and non-zero vectors x and y with $Ax = 2x$ and $Ay = 3y$, then x and y are orthogonal.

   true **false**

   (b) A $n \times n$ matrix A and A^T have the same eigenvectors.

   true **false**

   (c) For two matrices A, B, if the product AB is defined, then $(AB)^T = A^T B^T$.

   true **false**

   (d) If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then the span of $\{\vec{x}, \vec{y}\}$ is equal to the span of $\{\vec{x}, \vec{x} - \vec{y}\}$.

   true **false**

   (e) This is a subspace of $\mathbb{R}^3$: $H = \{\vec{x} \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 1\}$

   true **false**

   (f) For any matrix A, if $x \in \text{Col}A$, and $y \in \text{Null}A$, then $x^T y = 0$.

   true **false**

7. (4 points) Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**. You don't need to explain your reasoning.

   (a) Matrix A is 5×10, $b \in \mathbb{R}^5$, and $Ax = b$ has a unique solution.

   possible **impossible**

   (b) Matrix A has echelon form E, and $\text{Null}A \neq \text{Null}E$.

   possible **impossible**

   (c) Matrix A has a null space of dimension 1, and the linear transformation T_A is one to one.

   possible **impossible**

   (d) Matrix A is 3×4 and has orthonormal columns.

   possible **impossible**

You do not need to explain your reasoning for questions on this page.

8. (4 points) $H = \{\vec{x} \in \mathbb{R}^4 \ : \ x_1 = 5x_4\}$.
   (a) Write down a basis for H.

   (b) Write down a basis for $H^\perp$.

9. (4 points) Fill in the blanks.
   (a) Complete the matrix below so that the least squares solution to $Ax = b$ does not have a unique solution

$$\begin{pmatrix} 1 & \_\_\_ \\ 1 & \_\_\_ \\ 1 & \_\_\_ \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

   (b) For the system below, give an example of a choice of vector $\vec{b}$ for which the system is inconsistent.

$$\begin{pmatrix} 2 & 3 \\ 0 & 0 \\ 4 & 6 \end{pmatrix} \vec{x} = \vec{b} = \begin{pmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{pmatrix}$$

   (c) The dimension of the subspace of $\mathbb{R}^4$ spanned by the vectors below is ______.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -2 \\ -3 \\ 0 \end{pmatrix}$$

   (d) If $A = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 \end{pmatrix}$ has QR factorization $QR = \begin{pmatrix} \vec{q}_1 & \vec{q}_2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix}$, the length of $\vec{a}_2$ is ______.

You do not need to justify your reasoning for questions on this page.

10. (4 points) Below is a SVD factorization for a matrix $A = U\Sigma V^T$, where

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}, \ V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix}$$

Fill in the blanks.

(a) What is the rank of A? ________

(b) What is the largest value of $\|A\vec{x}\|$, subject to $\|\vec{x}\| = 1$? ________

(c) List an orthonormal basis for Null$A$. ____________________

(d) List an orthonormal basis for Col$A$. ____________________

11. (6 points) If possible, give an example of the following. If it is not possible, write "not possible".

(a) A matrix, A, that is in echelon form, and

$$\dim\left((\text{Row}(A))^\perp\right) = 3, \qquad \dim\left((\text{Col}(A))^\perp\right) = 1$$

(b) A 2×2 matrix in RREF, is diagonalizable, and is singular.

(c) A 2×3 matrix, A, in RREF, and Null(A) is spanned by $\vec{v} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$.

12. (4 points) Calculate the least squares solution, $\hat{x}$, to the equation below. Don't forget to show your work.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\hat{x} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}}$$

13. (2 points) What is the symmetric matrix A associated to the quadratic form below.

$$x_1^2 - 9x_2^2 - x_3^2 + 16x_1x_3$$

14. (2 points) S is the parallelogram determined by $\vec{v}_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, what is the area of the image of S under the map $\vec{x} \mapsto A\vec{x}$? Justify your reasoning.

15. (4 points) For what values of $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the system below consistent? Express your answer using parametric vector form. Justify your reasoning.

$$\begin{pmatrix} 0 & 4 \\ 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

16. (5 points) Consider the sequence of row operations that reduce matrix A to the identity.

$$A = \underbrace{\begin{pmatrix} -1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 4 & 1 \end{pmatrix}}_{A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix}}_{E_1A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix}}_{E_2E_1A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_3E_2E_1A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_4E_3E_2E_1A} = I_3$$

(i) Construct the four elementary matrices E_1, E_2, E_3, and E_4.

(ii) Consider the matrix products listed below. Which (if any) represents A, and which (if any) represents A^{-1}?

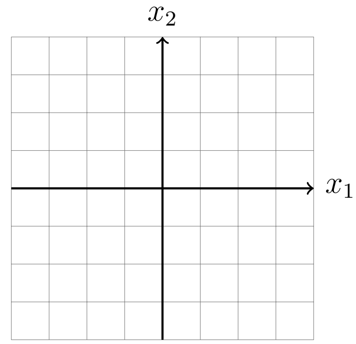
(a) $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$
(b) $E_4E_3E_2E_1$
(c) $E_1E_2E_3E_4$
(d) $E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$

17. (5 points) Let $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$.

(i) State the eigenvalues and eigenspaces of A.

(ii) Draw the eigenspaces of A and label them with the corresponding eigenvalue.

(a) eigenspaces



18. (4 points) If A is a matrix with independent columns, explain step by step how to find the QR factorization of A.

19. (3 points) Let $m > n$. Can n vectors span $\mathbb{R}^m$? Explain your reasoning.

20. (3 points) Let A be an $m \times n$ matrix. Explain why the matrix $A^T A$ has non-negative eigenvalues.

SOLUTIONS

Your initials: ______________

You do not need to explain your reasoning for questions on this page.

1. (6 points) Circle **true** if the statement is true, otherwise, circle **false.**

   (a) A product of invertible matrices is also invertible.

   true **false**

   (b) Regardless what A and $\vec{b}$ are, there is always at least one least-squares solution $\hat{x}$ to $A\vec{x} = \vec{b}$.

   true **false**

   (c) If $A\vec{x}_0 = \vec{b}$, and $A\vec{y} = \vec{0}$, then $\vec{x} = \vec{x}_0 - 5\vec{y}$ is a solution to $A\vec{x} = \vec{b}$.

   true **false**

   (d) An example of a quadratic form is the polynomial $7x_1^2 + 5x_2^2 - 10x_1x_2 + x_2$.

   true **false**

   (e) If a matrix is invertible then it is also diagonalizable.

   true **false**

   (f) A $n \times n$ matrix A and its echelon form E have the same eigenvalues.

   true **false**

2. (6 points) Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**.

   (a) The columns of matrix A are linearly independent, and NullA^T is not trivial.

   possible **impossible** eg $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

   (b) A is $n \times n$, $\lambda \in \mathbb{R}$ is an eigenvalue of A, and $\dim(\text{Col}(A - \lambda I)^{\perp}) = 0$.

   possible **impossible**

   (c) Stochastic matrix P has zero entries and is regular.

   possible **impossible**

   (d) A is a square matrix that is not diagonalizable, but A^2 is diagonalizable.

   possible **impossible** $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A^2 = 0_{2\times 2}$

   (e) A is 5×4, $A\vec{x} = \vec{b}$ has three free variables, and $\dim(\text{Row}(A)^{\perp}) = 3$.

   possible **impossible**

   (f) A $m \times n$ matrix A has linear transformation T_A. The map T_A can be one-to-one but not onto.

   possible **impossible** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

12

You do not need to explain your reasoning for questions on this page.

3. (8 points) If possible, give an example of the following. If it is not possible, write "not possible".

(a) A matrix that is 2×4, in reduced echelon form, with the dimension of column space being 3, and dimension of null space is 1.

1

not possible

(b) A 3×4 matrix with orthonormal columns.

1

not possible

(c) A 3×2 matrix A in reduced echelon form so that $A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2

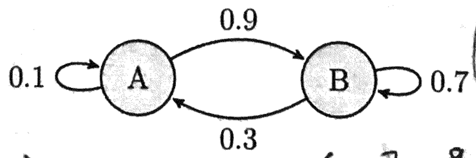
$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

1 3x2 and RREF

1 $A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(for $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ give 1 point out of 2)

(d) A stochastic matrix for the Markov Chain below.



$\left(\begin{array}{c|cc} & A & B \\ \hline A & .1 & .3 \\ B & .9 & .7 \end{array} \right.,\quad \begin{array}{c|cc} & B & A \\ \hline B & .7 & .9 \\ A & .3 & .1 \end{array} \Big)$

2

$P = \begin{pmatrix} .1 & .3 \\ .9 & .7 \end{pmatrix}$ or $P = \begin{pmatrix} .7 & .9 \\ .3 & .1 \end{pmatrix}$

1 2x2 and stochastic

1 Correct answer

(for $P = \begin{pmatrix} .3 & .1 \\ .7 & .9 \end{pmatrix}$ or $P = \begin{pmatrix} .9 & .7 \\ .1 & .3 \end{pmatrix}$ give 1 out of 2)

(e) A 2×2 matrix whose column space is the line $2x_1 + x_2 = 0$, and whose null space is the line $4x_1 - x_2 = 0$.

$\text{col} A = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} = \text{span}\left\{ \begin{pmatrix} 4 \\ -8 \end{pmatrix} \right\}$

2

$\Rightarrow A = \begin{pmatrix} 4 & -1 \\ -8 & +2 \end{pmatrix}$

1 Col A correct

1 Null A correct

(but $A = \begin{pmatrix} 1 & -4 \\ -2 & 8 \end{pmatrix}$ is ok, I guess) zero points for: any 2x2 matrix, or matrix with lin. indep. columns, or $\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$

3

4. (10 points) A has exactly two distinct eigenvalues, which are -2, and 1.

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

(a) Construct an eigenbasis for eigenvalue $\lambda = -2$.

$$A - (-2)I = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & 3 \\ 0 & 3 & 3 \\ 1 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

1 show work, and $A+2I$

1 answer

(b) Construct an eigenbasis for eigenvalue $\lambda = 1$.

$$A - (+1)I = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

1 show work, $A-I$

2 eigenvectors

(c) If possible, construct matrices P and D such that $A = PDP^T$, and P is a matrix with ~~orthogonal~~ orthonormal columns, D is diagonal.

$$\hat{v}_3 = \vec{v}_3 - \frac{v_2 \cdot v_3}{v_2 \cdot v_2} v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{-1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ +1/2 \\ 1 \end{pmatrix}$$

1 Gram Schmidt

can use $\frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$$\Rightarrow \quad P = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix}$$

1 normalizing

1 orthogonal matrix

$$D = \begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

1 diagonal 3x3

1 eigens match

(For distance (high school) exams, 5 points on (c) for $P = I$ and $D = A$.)

5. (10 points) Construct the singular value decomposition of $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

- $A^T A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ — (1) correct
- eigenvalues of $A^T A$ are, by inspection, $\lambda = 0, 1, 2$ — (1) all correct
- $\Sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (not $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$ and not $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$) — (1) Σ is 2x3, (1) elements, (−1) for no work shown to obtain Σ

V MATRIX

$\lambda_3 = 0$: $A^T A - 0I = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_2 = 1$: $A^T A - 1I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_1 = 2$: $A^T A - 2I = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(2) work shown and all correct (perfect) ((−1) for no work)

$\Rightarrow V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ +1 & 0 & -1 \end{pmatrix}$ (or $\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & +1 \end{pmatrix}}_{\text{also ok}}$) — (1) orthonormal columns and 3×3

U MATRIX

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(1) all correct ((−1) for no work)

$\Rightarrow U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, NOT $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ — (1) for U

$\Rightarrow A = U \Sigma V^T$, U, Σ, V as above — (1) this statement necessary.

II 1

15

6. (6 points) Circle **true** if the statement is true, otherwise, circle **false**. You do not need to explain your reasoning.

   (a) For any $n \times n$ matrix A, and non-zero vectors x and y with $Ax = 2x$ and $Ay = 3y$, then x and y are orthogonal.

   true **false**

   (b) A $n \times n$ matrix A and A^T have the same eigenvectors.

   true **false**

   (c) For two matrices A, B, if the product AB is defined, then $(AB)^T = A^T B^T$.

   true **false**

   (d) If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then the span of $\{\vec{x}, \vec{y}\}$ is equal to the span of $\{\vec{x}, \vec{x} - \vec{y}\}$.

   true **false**

   (e) This is a subspace of $\mathbb{R}^3$: $H = \{\vec{x} \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 1\}$

   true **false**

   (f) For any matrix A, if $x \in \text{Col}A$, and $y \in \text{Null}A$, then $x^T y = 0$.

   true **false**

7. (4 points) Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**. You don't need to explain your reasoning.

   (a) Matrix A is 5×10, $b \in \mathbb{R}^5$, and $Ax = b$ has a unique solution.

   possible **impossible**

   (b) Matrix A has echelon form E, and $\text{Null}A \neq \text{Null}E$.

   possible **impossible**

   (c) Matrix A has a null space of dimension 1, and the linear transformation T_A is one to one.

   possible **impossible**

   (d) Matrix A is 3×4 and has orthonormal columns.

   possible **impossible**

II 2

6

if student writes $\begin{pmatrix}5\\0\\0\\1\end{pmatrix}$ for H, and $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\5\end{pmatrix}$ for $H^\perp$, 3 points

You do not need to explain your reasoning for questions on this page.

8. (4 points) $H = \{\vec{x} \in \mathbb{R}^4 \ : \ x_1 = 5x_4\}$.

   (a) Write down a basis for H.

$$\begin{pmatrix}5\\0\\0\\1\end{pmatrix}, \begin{pmatrix}5k\\1\\0\\k\end{pmatrix}, \begin{pmatrix}5k\\0\\1\\k\end{pmatrix}$$

2

① a vector that is in H

① correct everything

(student can set k to be anything, most will use $k=0$)

   (b) Write down a basis for $H^\perp$.

$$\begin{pmatrix}1\\0\\0\\-5\end{pmatrix}$$

2

① exactly one vector in $\mathbb{R}^4$

① correct everything

(can also use $\begin{pmatrix}-1\\0\\0\\5\end{pmatrix}$)

9. (4 points) Fill in the blanks.

   (a) Complete the matrix below so that the least squares solution to $Ax = b$ does not have a unique solution

$$\begin{pmatrix}1 & k\\1 & k\\1 & k\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}4\\0\\1\end{pmatrix}$$

1

A must have linearly dependent columns

(k can be anything)

   (b) For the system below, give an example of a choice of vector $\vec{b}$ for which the system is inconsistent.

$$\begin{pmatrix}2 & 3\\0 & 0\\4 & 6\end{pmatrix}\vec{x} = \vec{b} = \begin{pmatrix}*\\1\\*\end{pmatrix}$$

1

← that element must not be equal to 0

($*$ = does not matter)

   (c) The dimension of the subspace of $\mathbb{R}^4$ spanned by the vectors below is 2.

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}1\\0\\0\\4\end{pmatrix}, \begin{pmatrix}0\\-2\\-3\\0\end{pmatrix}$$

1

   (d) If $A = (\vec{a}_1 \ \ \vec{a}_2)$ has QR factorization $QR = (\vec{q}_1 \ \ \vec{q}_2)\begin{pmatrix}2 & 4\\0 & 3\end{pmatrix}$, the length of $\vec{a}_2$ is 5.

1

II 3

17

You do not need to explain your reasoning for questions on this page.

5 10. (4 points) Below is a SVD factorization for a matrix $A = U\Sigma V^T$, where

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}, \ V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix}$$

Fill in the blanks.

1 (a) What is the rank of A? 3

1 (b) What is the largest value of $\|A\vec{x}\|$, subject to $\|\vec{x}\| = 1$? 5

1 (c) List an orthonormal basis for Null$A$. v_4, v_5

1 (d) List an orthonormal basis for Col$A$. u_1, u_2, u_3

no half points

6 11. (6 points) If possible, give an example of the following. If it is not possible, write "not possible".

(a) A matrix, A, that is in echelon form, and

$Col A^{\perp} = Null\ A^T$

$$\dim\left((\mathrm{Row}(A))^{\perp}\right) = 3, \qquad \dim\left((\mathrm{Col}(A))^{\perp}\right) = 1$$

2

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$, or $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

or $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, ...

(b) A 2×2 matrix in RREF, is diagonalizable, and is singular.

the only correct answers are $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

2

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable, give 1 point

(c) A 2×3 matrix, A, in RREF, and Null(A) is spanned by $\vec{v} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$.

$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \end{pmatrix}$

1 RREF and 2×3

2 two pivots and $\vec{v} \in Null\ A$

common error: $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & +4 \end{pmatrix}$ worth 1 point

A 7

12. (4 points) Calculate the least squares solution, $\hat{x}$, to the equation below. Don't forget to show your work.

$$Ax = b = \begin{array}{ccc} a_1 & a_2 & a_3 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\hat{x} = \begin{pmatrix} 0 \\ 1/4 \\ 1/2 \end{pmatrix}}$$

students don't actually have to fill this out

METHOD 1

Columns are orthogonal. ← (1) recognized can use dot products

$$\text{proj}_{\text{ColA}} b = \frac{b \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{b \cdot a_2}{a_2 \cdot a_2} a_2 + \frac{b \cdot a_3}{a_3 \cdot a_3} a_3$$

(1) used correct dot products

$$= 0\, a_1 + \frac{1}{4} a_2 + \frac{1}{2} a_3$$

(1) algebra

$$\Rightarrow \hat{x} = \begin{pmatrix} 0 \\ 1/4 \\ 1/2 \end{pmatrix}$$

(1) correct ans.

METHOD 2

$$A^T A \hat{x} = A^T b$$

(1) normal equations, correct

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \hat{x} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

(1) substitution

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \hat{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \hat{x} = \begin{pmatrix} 0 \\ 1/4 \\ 1/2 \end{pmatrix}$$

(2) algebra and correct answer.

METHOD 3: QR: I doubt many students will QR it.

METHOD 4: row reducing $Ax=b$ should get zero points.

19

8

13. (2 points) What is the symmetric matrix A associated to the quadratic form below.

$$x_1^2 - 9x_2^2 - x_3^2 + 16x_1x_3$$

$$A = \begin{pmatrix} 1 & 0 & 8 \\ 0 & -9 & 0 \\ 8 & 0 & -1 \end{pmatrix}$$

(1) main diagonal elements all correct

(1) off diag elements all correct

9

14. (2 points) S is the parallelogram determined by $\vec{v}_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, what is the area of the image of S under the map $\vec{x} \mapsto A\vec{x}$? Justify your reasoning.

area of S under map $= \left|\det A \det\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}\right| = |-10 \cdot 2| = 20$

or: $\left|\det\left(A \cdot \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}\right)\right| = \left|\det\begin{pmatrix} -4 & 3 \\ 4 & 2 \end{pmatrix}\right| = 20$

10

15. (4 points) For what values of $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the system below consistent? Express your answer using parametric vector form. Justify your reasoning.

$$\begin{pmatrix} 0 & 4 \\ 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 0 & 4 & b_1 \\ 1 & 3 & b_2 \\ 2 & 2 & b_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & 4 & b_1 \\ 1 & 3 & b_2 \\ 0 & -4 & b_3 - 2b_2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & b_2 \\ 0 & 4 & b_1 \\ 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right)$$

(1) knowing row reductions were necessary

(1) algebra to this step

$\Rightarrow b_3 - 2b_2 + b_1 = 0$

$\Rightarrow \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 2b_2 - b_3 \\ b_2 \\ b_3 \end{pmatrix} = b_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(1) something in parametric vector form

(1) correct

Some students do this:

$$\underbrace{\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{(1)} = \underbrace{x_1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}}_{(1)}$$

This doesn't answer the question, give at most two points.

Solutions

16) Consider the sequence of row operations that reduce matrix A to the identity.

$$A = \underbrace{\begin{pmatrix} -1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 4 & 1 \end{pmatrix}}_{A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix}}_{E_1 A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix}}_{E_2 E_1 A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_3 E_2 E_1 A} \sim \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_4 E_3 E_2 E_1 A} = I_3$$

(i) Construct the four elementary matrices E_1, E_2, E_3, and E_4.

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) Consider the matrix products listed below. Which (if any) represents A, and which (if any) represents A^{-1}?

i. $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$
ii. $E_4E_3E_2E_1$
iii. $E_1E_2E_3E_4$
iv. $E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$

$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$, and $A^{-1} = E_4E_3E_2E_1$.

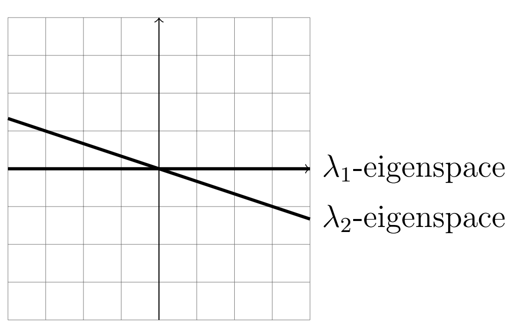
17) Let $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$.

(i) State the eigenvalues and eigenspaces of A.

$\lambda_1 = 2, \lambda_2 = 1$

λ_1-eigenspace is Span$\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$, λ_2-eigenspace is Span$\{\begin{pmatrix} 3 \\ -1 \end{pmatrix}\}$

(ii) Draw the eigenspaces of A and label them with the corresponding eigenvalue.



18) To create Q, Gram-Schmidt vectors, normalize each vector so they all have unit length, place vectors into matrix. To create R, compute $R = Q^T A$.

19) Place vectors into a matrix. The matrix will be $m \times n$. Because $n < m$, the matrix has at most n pivots. The dimension of the column space of the matrix is at most n, which means the vectors cannot span $\mathbb{R}^m$.

20) Let v_j be an eigenvector of **A^TA**

$$||Av_j|| = Av_j \cdot Av_j = v_j^T A^T A v_j = \lambda_j v_j \cdot v_j = \lambda_j \textbf{||vj||\^{}2}$$

Therefore all eigenvalues are positive or zero, but never negative.