Sample Final A, Math 1554

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

First Name ____________________ Last Name ____________________

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You do not need to justify your reasoning for questions on this page.

1. (10 points) Determine whether the statements are true or false.

   a) Every line in $\mathbb{R}^n$ is a one-dimensional subspace.

   b) If a quadratic form is indefinite, then the associated symmetric matrix is not invertible.

   c) If A is a diagonalizable $n \times n$ matrix, then rank$(A) = n$.

   d) If A is an orthogonal matrix, then the largest singular value of A is 1.

   e) If a linear system has more unknowns than equations, then the system cannot have a unique solution.

   f) If S is a one-dimensional subspace of $\mathbb{R}^2$, then so is $S^\perp$.

   g) If the columns of matrix A span $\mathbb{R}^m$, then the equation $A\vec{x} = \vec{b}$ is consistent for each $\vec{b}$ in $\mathbb{R}^m$.

   h) If A and B are square matrices and $AB = I$, then A is invertible.

   i) A steady state of a stochastic matrix is unique.

   j) The Gram-Schmidt algorithm applied to the columns of an $n \times n$ singular matrix produces a set of vectors that form a basis for $\mathbb{R}^n$.

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2. (10 points) Give an example of the following. If it is not possible to do so, write *not possible*.

   (a) A matrix $A \in \mathbb{R}^{2\times 2}$ that is in echelon form, is orthogonally diagonalizable, but is not invertible.

   $$A = \begin{pmatrix} & \\ & \end{pmatrix}$$

   (b) A negative semi-definite quadratic form, Q that has no cross terms and is expressed in the form $\vec{x}^T A \vec{x}$, where $\vec{x} \in \mathbb{R}^4$.

   $$Q =$$

   (c) A matrix, A, that is the standard matrix for the linear transform $T_A : \mathbb{R}^2 \to \mathbb{R}^2$. T_A first reflects points across the line $x_1 = x_2$, and then projects them onto the x_2-axis.

   $$A = \begin{pmatrix} & \\ & \end{pmatrix}$$

   (d) A matrix, A, that is in echelon form, is 3×4, with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$. The first two columns of the matrix, $\vec{a}_1$ and $\vec{a}_2$, are linearly independent vectors. Vectors $\vec{a}_3$ and $\vec{a}_4$ are in Span $\{\vec{a}_1, \vec{a}_2\}$.

   $$A = \begin{pmatrix} & \\ & \end{pmatrix}$$

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3. (10 points) If possible, give an example of the following. If it is not possible, write "not possible".

   (a) A 5×3 matrix, X, in RREF, such that dim(Col(X)) $= 2$, and dim(Null(X)) $= 3$.

   (b) A 3×3 matrix, Y, in RREF, Row$(Y)^{\perp}$ is spanned by $\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$.

   (c) A 3×3 matrix, Z, that is not diagonalizable. Z is singular and upper triangular.

   (d) A matrix that has one eigenvalue, $\lambda = 3$. The eigenvalue λ has algebraic multiplicity 2, and geometric multiplicity 2.

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4. (5 points) Fill in the blanks.
   (a) The dimension of the null space of an $n \times n$ invertible matrix is __________.
   (b) The rank of a 4×5 matrix whose null space is 3-dimensional is __________.
   (c) Matrix A has two distinct eigenvalues: eigenvalue $\lambda_1 = -2$ with algebraic multiplicity 3, and eigenvalue $\lambda_2 = 3$ with algebraic multiplicity 1.
      1. The characteristic equation of A is ______________________________.
      2. The dimensions of A are _______________.
      3. The nullity of A is equal to _______________.

5. (3 points) Suppose T_A is an onto linear transformation. Circle the matrices that A could be equal to, if any.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

6. (2 points) Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**. You don't need to explain your reasoning.
   (a) $\vec{v}$ is a non-zero vector in $\mathbb{R}^3$, W is a non-empty subspace of $\mathbb{R}^3$, and $(\text{proj}_W\vec{v}) \cdot \vec{v} = \vec{0}$.

   possible **impossible**

   (b) A is 2×3, $\dim(\text{Col}(A))^{\perp} = 1$, and A has one pivot column.

   possible **impossible**

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7. (4 points) A, B, and C are $n \times n$ invertible matrices. Construct expressions for X and Y in terms of A, B, and C. Don't forget to justify your reasoning.

$$\begin{pmatrix} X & 0 & 0 \\ Y & 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A \\ B & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$\boxed{X =}$

$\boxed{Y =}$

8. (6 points) Solve the equation $A\vec{x} = \vec{b}$ by using the LU factorization of A. Do not solve the system by computing A.

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

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9. (10 points) Matrix A has only two distinct eigenvalues, which are 5 and -3.

$$A = \begin{pmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{pmatrix}$$

.

(a) Construct a basis for the eigenspace of A associated with $\lambda_1 = 5$.

(b) Construct a basis for the eigenspace of A associated with $\lambda_2 = -3$.

(c) If possible, construct matrices P and D such that $A = PDP^{-1}$.

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10. (10 points) Let $A = QR$ be as below.

$$A = QR = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{2} \\ 1 & 0 \\ 1 & 0 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 0 & 2\sqrt{2} \end{bmatrix}$$

(a) $\dim(\mathrm{Null}(Q)) =$ __________

(b) The length of the first column of A is __________

(c) Give an orthogonal basis for $\mathrm{Col}(A)$.

$$\boxed{\text{basis} = \left\{ \qquad\qquad\qquad \right\}}$$

(d) Determine the least-squares solution to $A\hat{x} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

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11. (10 points) Suppose $Q(\vec{x}) = 2x_1^2 + 6x_1x_2 - 6x_2^2, \quad \vec{x} \in \mathbb{R}^2$.

    i) Make a change of variable, $\vec{x} = P\vec{y}$, that transforms the quadratic form Q into one that does not have cross-product terms. Give P and the new quadratic form.

    ii) Answer the following. You do not need to justify your reasoning.

    (a) Classify the quadratic form (e.g. positive definite, positive semidefinite).

    (b) State the largest value of Q subject to $||\vec{x}|| = 1$.

    (c) What is the maximum value of Q, subject to the constraints, $\vec{x}\cdot\vec{u} = 0$ and $||\vec{x}|| = 1$?

    (d) Give a vector, $\vec{u}$, that specifies a location where the largest value of Q, subject to $||\vec{x}|| = 1$, is obtained.

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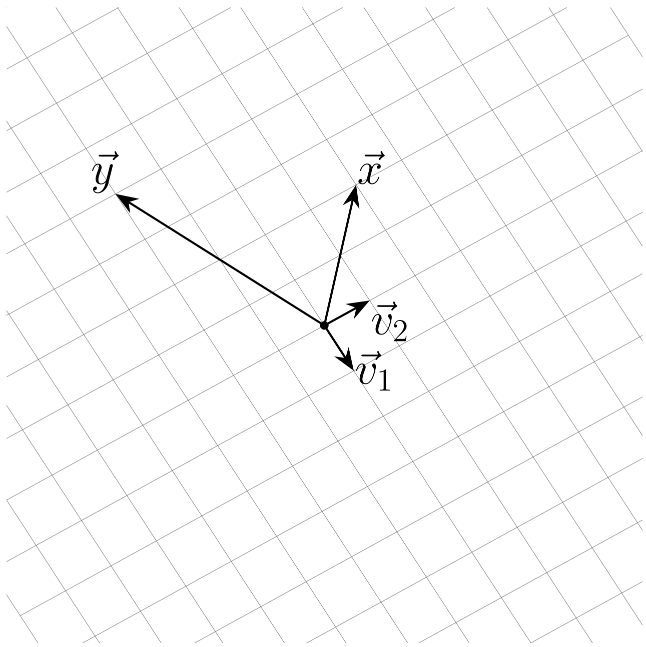
12. (6 points) Let $A = P \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}$, where P has columns $\vec{v}_1$ and $\vec{v}_2$.

   (i) State the eigenvalues of A.

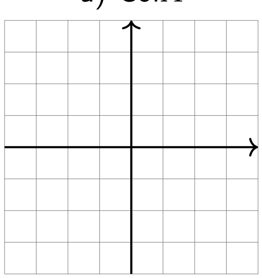
   (ii) Draw $A\vec{x}$ and $A\vec{y}$.



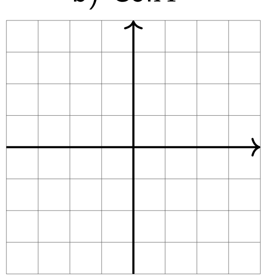
   (iii) State a non-zero vector $\vec{p} \in \mathbb{R}^2$ such that $A^k\vec{p} \to \vec{0}$ as $k \to \infty$.

13. (4 points) Let $A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$. Sketch a) Col$A$, b) Col$A^\perp$, c) Row$A^\perp$, and d) Null$A^\perp$.

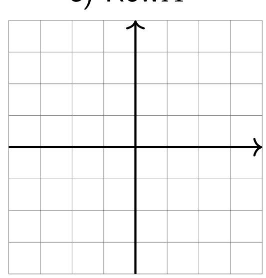
a) Col$A$



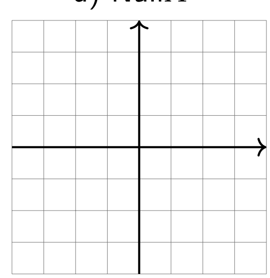
b) Col$A^\perp$



c) Row$A^\perp$



d) Null$A^\perp$



Sample Final A, Answers

1. True/false.
   (a) False. For example $y = x + 1$
   (b) False. For example $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
   (c) False.
   (d) True
   (e) True
   (f) True
   (g) True
   (h) True
   (i) False
   (j) False

2. Example construction I.
   (a) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
   (b) $\vec{x}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \vec{x}$
   (c) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
   (d) $\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$, where $*$ is arbitrary

3. Example construction II.
   (a) Not possible.
   (b) $\begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}$
   (c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
   (d) $3I_2$, or $3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4. Fill in the blank (FITB).
   (a) 0
   (b) 2
   (c) $0 = (\lambda + 2)^3(\lambda - 3)$
   (d) 4×4
   (e) 0
5. Only the first matrix.
6. Possible/impossible.
   (a) possible
   (b) possible
7. Solve for X:

$$\begin{aligned} XA &= I \\ XAA^{-1} &= IA^{-1} \\ X &= A^{-1} \end{aligned}$$

Now solve for Y.

$$\begin{aligned} YA + 0 + IB &= 0 \\ YA &= -B \\ YAA^{-1} &= -BA^{-1} \\ Y &= -BA^{-1} \end{aligned}$$

For full points show some work.

8. Let $\vec{y} = U\vec{x}$. Then $L\vec{y} = \vec{b}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

By inspection $y_1 = y_2 = 1$ and $y_3 = 2$. Now solve $U\vec{x} = \vec{y}$.

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 : x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

By inspection,

$$\begin{aligned} x_3 &= 2 - 2x_4 \\ x_2 &= 1 - 4x_4 \\ x_1 &= 1 + x_4 - x_3 = -1 + 3x_4 \end{aligned}$$

Thus,

$$\vec{x} = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ -4 \\ -2 \\ 1 \end{pmatrix}$$

9. Diagonalization.

   (a)

   $$A - 5I = \begin{pmatrix} -12 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & 4 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   Thus $3x_1 + 4x_2 - x_3 = 0$. We obtain two eigenvectors,

   $$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$$

   (b)

   $$A+3I = \begin{pmatrix} -4 & -16 & 4 \\ 6 & 16 & -2 \\ 12 & 16 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -1 \\ 3 & 8 & -1 \\ 3 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -1 \\ 0 & -4 & 2 \\ 0 & -4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

   $$\Rightarrow \vec{v}_3 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

   (c) Place eigenvectors and eigenvalues into P and D.

   $$P = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -3 & -1 \\ 3 & 0 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & & \\ & 5 & \\ & & -3 \end{pmatrix}$$

10. (a) 0

    (b) 6

    (c) The columns of Q, which are $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ 0 \\ 0 \\ \sqrt{2} \end{pmatrix}$

    (d)

    $$R\hat{x} = Q^T\vec{b}$$

    $$\begin{pmatrix} 6 & 2 \\ 0 & 2\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -\sqrt{2} & 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

    $$\begin{pmatrix} 6 & 2 \\ 0 & 2\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So $x_1 = 1/6$ and $x_2 = 0$.

11.

$$Q = x^T A x = x^T \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} x$$

By inspection $\lambda = -7, 3$. Could solve $0 = \det(A - \lambda I)$. For $\lambda_1 = -7$,

$$A + 7I = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \Rightarrow \vec{v}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

For $\lambda_2 = 3$,

$$A - 3I = \begin{pmatrix} -1 & 3 \\ * & * \end{pmatrix} \Rightarrow \vec{v}_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

We could also obtain $\vec{v}_2$ knowing that A is symmetric, so $\vec{v}_1$ and $\vec{v}_2$ must be orthogonal. But we can now construct P:

$$P = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$$

The new quadratic form is

$$Q = -7y_1^2 + 3y_2^2$$

The change of variable is $\vec{y} = P^{-1}\vec{x}$, or $\vec{x} = P\vec{y}$.

(i) indefinite

(ii) the largest eigenvalue, $\lambda_2 = 3$

(iii) other eigenvalue, $\lambda_1 = -7$

(iv) the unit eigenvector corresponding to the largest eigenvalue, which is $\vec{u} = \vec{v}_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ or $\vec{u} = -\vec{v}_2 = -\frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

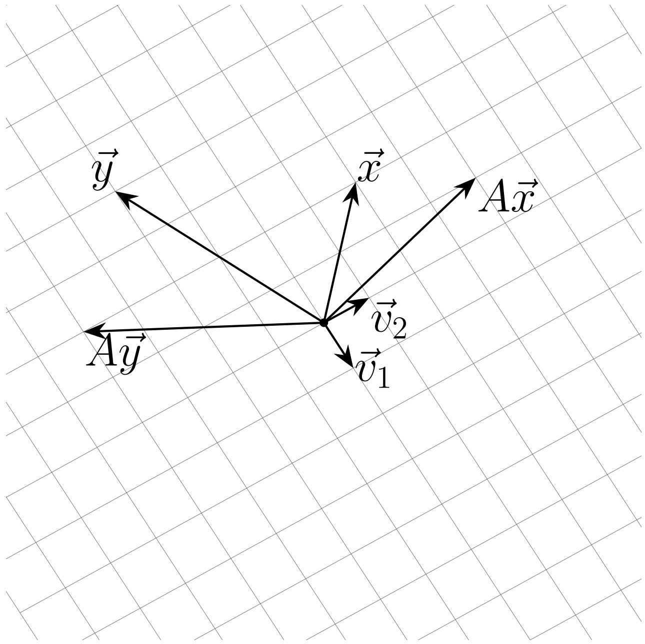
12. $A = P \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}$, where P has columns $\vec{v}_1$ and $\vec{v}_2$.

(i) By inspection, $\lambda_1 = 1/2$, $\lambda_2 = 2$

(ii)

$$Ax = A(2v_2 - 2v_1) = 2Av_2 - 2Av_1 = 4v_2 - v_1$$

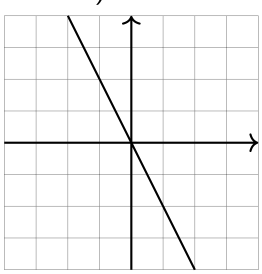
$$Ay = A(-2v_2 - 4v_1) = -4v_2 - 2v_1$$



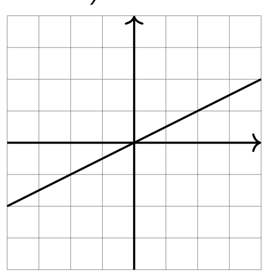
(iii) If $p = v_1$, then $A^k p = \lambda_1^k v_1 \to \vec{0}$ as $k \to \infty$ because $\lambda_1^k \to 0$.

13. Let $A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$. Sketch a) Col$A$, b) Col$A^\perp$, c) Row$A^\perp$, and d) Null$A^\perp$.

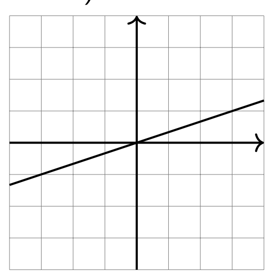
a) Col$A$



b) Col$A^\perp$



c) Row$A^\perp$



d) Null$A^\perp$