Additional Problems for Final Exam Review

This is a set of questions that students can use to help them prepare for the final exam.

About this Review Set

As stated in the syllabus, a primary goal of this course is to prepare students for more advanced courses that have this course as a pre-requisite. **To help us meet this goal, solutions are not are provided for the additional review problem sets. This is intentional: upper level courses often don't have recitations, MML, and solutions for everything**. Students need to develop and use various strategies to check their solutions in those courses. In this course, students are encouraged to ask questions they may have about the course on Piazza, office hours, by checking their answers with their peers, or by asking their instructor after class. Calculators and software are also great ways to check your work. All of these methods are valuable skills that are transferable to higher level courses, and beyond.

Contents

1 True/False Exercises

Indicate whether each statement is true or false.

1.1) For any real $n \times n$ matrices A and B, $A \neq B$, $\text{rank}(A) + \text{rank}(B) = \text{rank}(A + B)$.

1.2) If $\vec{x} \in \text{Null}A$ then $\vec{x} \in \text{Null}A^T A$.

1.3) All elementary matrices are square and invertible.

1.4) If A is a symmetric matrix, then A^2 is a symmetric matrix.

1.5) If A is a square matrix, then A^2 is a symmetric matrix.

1.6) If A is any matrix, then $A^T A$ is a symmetric matrix.

1.7) An invertible matrix can have a zero singular value.

1.8) If $A \in \mathbb{R}^{m \times n}$, then

  (a) the eigenvalues of $A^T A$ are real.
  (b) the eigenvalues of $A^T A$ are non-negative.
  (c) the eigenvalues of $A^T A$ are positive.

1.9) If every coefficient in a quadratic form is positive, then the form is positive semi-definite.

1.10) If D is a square diagonal matrix, then any matrix of the form PDP^T is a symmetric matrix.

1.11) For a symmetric matrix, the dimension of the eigenspace always equals it algebraic dimension.

1.12) Let $\|\vec{u}\| = 1$, and $A = \vec{u}\vec{u}^T$. Then, A is symmetric.

1.13) Same A as above: $A^2 = A$.

1.14) For symmetric A and any vectors $\vec{x}$ and $\vec{y}$, $(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot A\vec{y}$.

1.15) It is possible for a quadratic form $Q(\vec{x})$, constrained to $||\vec{x}|| = 1$, to obtain a maximum value of $Q = C \geq 0$ at two distinct points, $\vec{x}_1$ and $\vec{x}_2$.

1.16) If A is a 3×3 matrix and has fewer than 3 pivots, then $A\vec{x} = \vec{b}$ must have infinitely many solutions.

1.17) A must be a square matrix for $A^T A$ to be defined.

1.18) If a 3×7 matrix has 2 pivot columns, then the dimension of the null space of that matrix is equal to 5.

1.19) If $\mathcal{B}$ is any basis for subspace H, then any vector in H can be represented as a linear combination of the vectors in $\mathcal{B}$.

1.20) If three row exchanges are made to a square matrix A to produce B, then det(A) = det(B).

1.21) If $A \in \mathbb{R}^{2\times 2}$ has complex eigenvalues λ_1 and λ_2, then $|\lambda_1| = |\lambda_2|$.

1.22) Matrices with the same eigenvalues are similar matrices.

1.23) If A is invertible and diagonalizable, then A^{-1} is diagonalizable.

1.24) Any matrix that is similar to the identity matrix must be equal to the identity matrix.

1.25) The $n \times n$ zero matrix can be diagonalized.

1.26) The set of all probability vectors in $\mathbb{R}^n$ forms a subspace of $\mathbb{R}^n$.

1.27) The set of eigenvectors of an $n \times n$ matrix, that are associated with an eigenvalue, λ, span a subspace of $\mathbb{R}^n$.

1.28) An eigenspace is a subspace spanned by a single eigenvector.

1.29) If A is $n \times n$ and A has n distinct eigenvalues, then the eigenvectors of A span $\mathbb{R}^n$.

1.30) $V = \{\vec{x} \in \mathbb{R}^4 \mid x_1 - x_2 = 0, x_4 = 1\}$ is a subspace.

1.31) All upper triangular matrices are in echelon form.

1.32) If a matrix has an LU factoriztion, it must be invertible.

1.33) If E_1 and E_2 are elementary matrices, then $E_1E_2 = E_2E_1$.

1.34) If E is an echelon form of A, then $\text{Col}(A)$ is equal to $\text{Col}(E)$.

1.35) If a matrix is row equivalent to the identity matrix, it cannot be singular.

1.36) If A is a 3×2 matrix with 2 pivot columns, $A\vec{x} = \vec{0}$ has a unique solution.

1.37) If a linear system has more unknowns than equations, then the system has either no solutions or infinitely many solutions.

1.38) If the span of the columns of a matrix A is a line, then $A\vec{x} = \vec{0}$ has an infinite number of solutions.

1.39) If A is a 4×7 matrix with four pivots, the columns of A span $\mathbb{R}^4$.

1.40) If the 3×3 matrix A has linearly dependent columns, there cannot be a pivot in every column of A.

1.41) If $\vec{a}$ and $\vec{b}$ are linearly independent, and $\vec{b}$ and $\vec{c}$ are linearly independent then vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ must be linearly independent.

1.42) If square matrix A is not invertible, then A cannot be diagonalized.

1.43) If a square matrix is stochastic, then at least one of its eigenvalues is equal to 1.

1.44) If a square matrix is upper triangular, then it cannot be regular stochastic.

1.45) An eigenspace is a subspace spanned by a single eigenvector.

1.46) Elements of a stochastic matrix can be zero.

1.47) Eigenvalues can be equal to zero, but eigenvectors cannot be zero vectors.

1.48) If $\vec{y}$ is in subspace $W^{\perp}$, then $\text{proj}_W\vec{y} = \vec{0}$.

1.49) If $\vec{x}$ is in subspace W, and $\vec{y} \in W^{\perp}$, then $\vec{x}^T\vec{y} = 0$.

1.50) The dimension of the null space of an orthogonal matrix is always equal to 0.

1.51) If A is an orthogonal matrix, then the linear transform $\vec{x} \to A\vec{x}$ preserves lengths.

1.52) For any vector $\vec{y} \in \mathbb{R}^2$ and subspace W, the vector $\vec{v} = \vec{y} - \text{proj}_W\vec{y}$ is orthogonal to W.

1.53) If A is a 2×2 matrix, then vectors in Col$A$ are orthogonal to vectors in Null$A$.

1.54) Every elementary matrix is invertible.

1.55) If $A\vec{x} = \vec{b}$ does not have any free variables and A is square, then A must be singular.

1.56) Row operations do not change the null space of a matrix.

1.57) If $\det(A) = 0$, then the only solution to $A\vec{x} = \vec{0}$ is the trivial solution.

1.58) If B, C, D are square invertible matrices and $BC = BD$, then $C = D$.

1.59) A basis for the column space of any square matrix is unique.

1.60) Any matrix A can be reduced to the identity matrix by multiplying A by a set of elementary matrices.

1.61) If A has the LU factorization $A = LU$ and L has exactly 3 pivot columns, then A also has exactly 3 pivot columns.

1.62) If $\vec{y}$ is in subspace W, then the projection of $\vec{y}$ onto W is $\vec{y}$.

1.63) If a matrix is square and orthogonal then it must also be invertible.

1.64) A basis for $(\text{Col}A)^{\perp}$ can be constructed by applying the Gram-Schmidt algorithm to the columns of A.

1.65) Every stochastic matrix has at least one steady state vector.

1.66) A stochastic matrix could have two distinct steady state vectors.

1.67) If A is symmetric, then the eigenvalues of A are real and non-negative.

1.68) If a 3×3 matrix has eigenvalues $\lambda = -3, -1, 2$, then its singular values are $\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1$.

1.69) If A is a 3×2 matrix with 2 pivot columns, $A\vec{x} = \vec{0}$ has a unique solution.

1.70) If a linear system has more unknowns than equations, then the system always has at least one solution.

1.71) If A has linearly dependent columns, then the columns of A span $\mathbb{R}^m$.

1.72) If A is a 6×7 matrix and has two pivot columns, $A\vec{x} = \vec{b}$ has 4 free variables.

1.73) If $A\vec{x} = \vec{0}$ and A has reduced echelon form E, then $E\vec{x} = \vec{0}$.

1.74) If every column of A is pivotal, then $A\vec{x} = \vec{b}$ is consistent for any $\vec{b}$.

1.75) The echelon form of A is unique.

1.76) If A is 5×7 and has two pivot columns, then $A\vec{x} = \vec{b}$ has 3 free variables.

1.77) If A has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^m$.

1.78) The points $(2, 8, 4, 2)$ and $(4, 16, 8, 4)$ are homogeneous coordinates for the same point in $\mathbb{R}^3$.

1.79) For any nonzero $\vec{v}$, the matrix $\vec{v}\vec{v}^T$ is the standard matrix for orthogonal projection on the line spanned by $\vec{v}$.

1.80) If A is symmetric, then the singular value decomposition of A is the same as eigenvalue decomposition of A.

1.81) If A is negative definite, then it does not have singular value decomposition.

1.82) The points $(2, 8, 4, 2)$ and $(4, 16, 8, 2)$ are homogeneous coordinates for the same point in $\mathbb{R}^3$.

1.83) The sum of two symmetric matrices is a symmetric matrix.

2 Example Construction Exercises

If possible, give an example of the following.

2.1) An indefinite quadratic form that has no cross terms and is expressed in the form $\vec{x}^T A\vec{x}$, where $\vec{x} \in \mathbb{R}^3$.

2.2) A non-zero matrix A whose SVD, $A = U\Sigma V^T$ has the property that $U = V$.

2.3) A matrix that does not have an SVD.

2.4) A non-zero 2×2 matrix, A, that has an SVD factorization, a QR factorization, an LU factorization, can be diagonalized as PDP^T, and $A \neq I_2$.

2.5) A non-zero 2×2 elementary matrix, A, that can be diagonalized as PDP^T, and $A \neq I_2$.

2.6) A real matrix whose null space is spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, and whose column space is spanned by $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

2.7) A real matrix A that is in RREF. Col(A) is spanned by u and v, Null(A) is spanned by x and y. *Hint: it is possible to construct such a matrix, and the solution is unique.*

$$u = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \; v = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \; x = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix}.$$

2.8) A real matrix with two distinct eigenvalues, λ_1 and λ_2. Eigenvalue λ_1 has algebraic multiplicity 3 and geometric multiplicity 2. λ_2 has algebraic multiplicity 1 and geometric multiplicity 1.

2.9) A characteristic polynomial for a matrix that has one distinct eigenvalue, $\lambda_1 = 4$. The algebraic multiplicity of the eigenvalue is equal to 2017.

2.10) A Markov chain that has a steady-state equal to $\vec{q} = \begin{pmatrix} .5 \\ .5 \end{pmatrix}$.

2.11) A 3×3 matrix A, that is in reduced echelon form, has exactly two pivot columns, and $A\vec{x} = \vec{0}$, where $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

2.12) A 2×6 matrix, A, in reduced echelon form, such that dim(Col(A)) = 3, and dim(Null(A)) = 3.

2.13) A 3×4 matrix, in reduced echelon form, and whose null space is spanned by the vector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

2.14) A 3×3 upper triangular matrix, whose null space is the plane $x_1 = x_2 + x_3$, and whose column space is spanned by $\vec{e}_1$.

2.15) A 2×2 regular stochastic matrix that is also singular.

2.16) A 2×2 singular matrix, A, that has a complex eigenvalue equal to $4i$.

2.17) A 2×2 rotation-dilation matrix that rotates vectors by $\pi/2$ radians and scales them by a factor of 4.

2.18) A vector $\vec{u} \in \mathbb{R}^3$ such that $\text{proj}_{\vec{p}}\,\vec{u} = \vec{p}$, where $\vec{u} \neq \vec{p}$, and $\vec{p} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

2.19) A matrix C in echelon form. The linear system $C\vec{x} = \vec{b}$ is inconsistent and does not have a unique least-squares solution, where $\vec{x} \in \mathbb{R}^2$ and
$\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

2.20) A matrix C, that is in reduced echelon form, and satisfies

$$\dim\left((\text{Row}(C))^{\perp}\right) = 4, \qquad \dim\left((\text{Col}(C))^{\perp}\right) = 1$$

2.21) A positive semi-definite quadratic form that has no cross terms, and is expressed in the form $\vec{x}^T A\vec{x}$, where $\vec{x} \in \mathbb{R}^3$.

2.22) An upper triangular matrix $A \in \mathbb{R}^{2\times 2}$ that is diagonalizable, but is not orthogonally diagonalizable.

2.23) A real 2×2 matrix, A, such that $T(\vec{x}) = A\vec{x}$ orthogonally projects $\vec{x} \in \mathbb{R}^2$ onto the line $x_2 = -x_1$.

2.24) A 3×3 matrix that corresponds to the composite transform of a scaling by a factor of 0.3, a rotation of $90°$ about the origin (counterclockwise), and finally a translation that adds $(-0.5, 2)$ to the origin.

3 Multiple Choice and Fill in the Blank Exercises

3.1) Fill in the blanks.

3.1.1) If S is a subspace of $\mathbb{R}^{90}$ and dim(S) = 48, then the dimension of the orthogonal compliment to S is: $\boxed{}$.

3.1.2) If $\vec{x} \to A\vec{x}$ is a one-to-one linear transform and A is 61×21, then the dimension of the orthogonal compliment of Row(A) is $\boxed{}$.

3.1.3) If $\vec{u}$ is a vector in $\mathbb{R}^n$ and $\vec{u} \cdot \vec{u} = \vec{0}$, then $\vec{u}$ must be equal to $\boxed{}$.

3.1.4) If A is 10×15 and dim(Null(A)$^\perp$) = 9, the rank of A is $\boxed{}$.

3.1.5) If A is a 20×60 matrix, then $\dim((\text{Row}A)^\perp) + \dim(\text{Col}A)$ is equal to $\boxed{}$.

3.1.6) If A is an $n \times n$ orthogonal matrix and $\vec{x}$ nd $\vec{y}$ are orthogonal vectors in $\mathbb{R}^n$, then $(A\vec{x}) \cdot (A\vec{y})$ is equal to $\boxed{}$.

3.1.7) If $x \to Ax$ is a one-to-one linear transform and A is 6×5, then the dimension of Col$(A)^\perp$ is $\boxed{}$.

3.1.8) The distance between $\vec{y} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and the line spanned by $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\boxed{}$.

3.1.9) An eigenvector of $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ is $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$.
What is the eigenvalue associated with eigenvector $\vec{v}_1$?

$$\boxed{}$$

3.1.10) For what values of k (if any) does $A = \begin{pmatrix} -2 & k \\ -1 & 0 \end{pmatrix}$ have exactly two distinct real eigenvalues?

$$\boxed{}$$

3.1.11) For what values of k (if any) is $A = \begin{pmatrix} 2 & 0 \\ k & 2 \end{pmatrix}$ diagonalizable?

$$\boxed{}$$

3.1.12) The characteristic polynomial of A is $(\lambda - 1)^2(\lambda - 3)\lambda^6$.

- What is the algebraic multiplicity of the eigenvalue $\lambda = 1$?
- What are the dimensions of matrix A?
- What is the value of $\det(A)$?

3.1.13) If A is a 5×11 matrix such that $\text{rank}(A) = 3$. What is the dimension of the nullspace of A?

3.1.14) List all possible values of x and y so that $A^2 = A$, where $A = \begin{pmatrix} x & 1 \\ y & 2 \end{pmatrix}$.

3.1.15) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$. What is the value of $\det(A)$?

3.1.16) B and C are square matrices with $\det(BC) = -2$ and $\det(C) = 3$. What is the value of $\det(B)\det(C^3)$?

3.1.17) A is a 3×2 matrix in RREF whose column space has dimension 2. How many different matrices can you construct that meet these criteria?

3.1.18) Suppose

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

What is the value of $\det(A)$?

3.1.19) Suppose

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & k \end{pmatrix}$$

For what values of k (if any) is A singular?

3.2) Some of the following phrases and expressions are not defined. Some of them are defined. Indicate which of the following are undefined.

(a) dim(A), for $n \times n$ matrix A.

(b) The dimension of a matrix A is 5.

(c) The dimension of a subspace S is 5.

(d) The dimensions of a matrix are 5×5.

(e) Subspace S has dimensions 5×5.

(f) Matrix A is linearly independent.

(g) $\dfrac{1}{\det(A)}$, for singular matrix A.

(h) $\dfrac{1}{A}$, for invertible matrix A.

(i) $\text{proj}_W(\vec{u} \cdot \vec{u})$, for $\vec{u}$ in $\mathbb{R}^3$, W is a subspace of $\mathbb{R}^3$.

(j) $(\text{proj}_W \vec{u}) \cdot \vec{u}$, for $\vec{u}$ in $\mathbb{R}^2$, W is a subspace of $\mathbb{R}^2$.

(k) $\text{proj}_W(\text{proj}_W \vec{u})$, for $\vec{u}$ in $\mathbb{R}^n$, W is a subspace of $\mathbb{R}^n$.

(l) $\vec{u} \cdot \vec{u} \cdot \vec{u}$, for $\vec{u}$ in $\mathbb{R}^n$.

(m) $\det(\det A)$, for A in $\mathbb{R}^{n \times n}$.

(n) $A^\perp$ for $m \times n$ matrix A.

3.3) For the following questions, construct an expression for the SVD of A.

(a) If $A = U\Sigma V^T$ is invertible, what is the SVD of A^{-1}? __________

(b) If $A = U\Sigma V^T$, what is the SVD of A^T? __________

(c) If $A = U\Sigma V^T$, state the largest value of $\|A\vec{x}\|$ subject to $\|\vec{x}\| = 1$. __________

4 Computation Exercises

4.1) Diagonalize the symmetric matrix, using an orthogonal matrix to do so. The eigenvalues and corresponding eigenvectors are given.

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$\lambda = -2, 7, 7$$

$$\lambda = -2 : \vec{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\lambda = 7 : \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$

4.2) Construct the matrix of the quadratic form $Q = 4x_1^2 - 2x_2x_3$, and classify the quadratic form. Assume $\vec{x} \in \mathbb{R}^3$.

4.3) Use a change of variable to express the quadratic form $Q = -5x_1^2 + 4x_1x_2 - 2x_2^2$ with no cross-product terms, and classify the quadratic form.

4.4) Suppose $Q = 7x_1^2 + x_2^2 + 7x_3^2 - 8x_1x_2 - 4x_1x_3 - 8x_2x_3$.

(a) Construct a matrix A so that Q can be expressed in the form $\vec{x}^T A\vec{x}$.

(b) Identify a location, $\vec{u}_1$, where the maximum value of Q is obtained, subject to the constraint $||\vec{x}|| = 1$. Hint: the eigenvalues of A are 9 and -3.

(c) Identify a location, $\vec{u}_2$, where the maximum value of Q is obtained, subject to the constraints $||\vec{x}|| = 1$ and $\vec{u}_1 \cdot \vec{u}_2 = 0$.

4.5) Compute the SVD for the two matrices below.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -3 & 0 \\ -7 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 1 & -2 \end{bmatrix}$$

4.6) Construct the SVD for $A = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$ and use your results to:

(a) determine the rank of A

(b) construct orthonormal bases for Col$A$, (Col$A$)$^\perp$, Null$A$, (Null$A$)$^\perp$.

4.7) Construct the SVD of A.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}$$

4.8) Suppose A is an invertible $n \times n$ matrix and $\vec{v}$ is an eigenvector of A with associated eigenvalue $\lambda = 6$. Determine one eigenvalue of the following matrices, if possible.

(a) A^9

(b) A^{-1}

(c) $A - 7I_n$

(d) $2A$

4.9) Matrix A is a 2×2 matrix whose eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$, and whose corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Calculate

(a) $A(\vec{v}_1 + 4\vec{v}_2)$

(b) A^{10}

(c) $\lim_{k\to\infty} A^k(\vec{v}_1 + 4\vec{v}_2)$

4.10) If the eigenvalues of symmetric 3×3 matrix A are $-2, 0$, and 3, compute the maximum and minimum values of Q subject to $||\vec{x}|| = 1$ and $\vec{x} \cdot \vec{u} = 0$. Give a location where the maximum value is obtained.

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = x_2^2 + 2x_1x_2^2 + 4x_1x_3 + 2x_2x_3, \quad \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

4.11) If the determinant $\left|\begin{pmatrix} a & b \\ 2 & 0 \end{pmatrix}\right| = -5$, compute the value of $\left|\begin{pmatrix} 1 & 0 \\ 3a & 3b \end{pmatrix}\right|$.

4.12) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Construct a matrix A so that $T(\vec{x}) = A\vec{x}$ for all vectors $\vec{x}$.

4.13) Below is a SVD factorization for a matrix A.

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 & \vec{u}_5 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}$$

(a) What is the rank of A?

(b) What is the largest value of $\|A\vec{x}\|$, subject to $\|\vec{x}\| = 1$?

(c) List an orthonormal basis for Nul$A$.

(d) List an orthonormal basis for Col$A$.

4.14) Below is a SVD factorization for a matrix A.

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \ V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$$

(a) What is the condition number of A?

(b) State the spectral decomposition of A.

(c) State an orthonormal basis for Col$A$.

4.15) Suppose $Q(\vec{x}) = x_1^2 + 4x_2^2 + x_3^2 + 4x_1x_2$, where $\vec{x} \in \mathbb{R}^3$.

(a) Compute the largest value of Q subject to $||\vec{x}|| = 1$.

(b) Give a vector, $\vec{u}$, that specifies a location where the largest value of Q, subject to $||\vec{x}|| = 1$, is obtained.

(c) What is the maximum value of Q, subject to the constraints, $\vec{x} \cdot \vec{u} = 0$ and $||\vec{x}|| = 1$?

4.16) A is a 2×2 matrix whose nullspace is the line $x_1 = x_2$, and

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Construct a basis for

(a) the nullspace of $X = AB$.

(b) the nullspace of $Y = AC$.

5 Other Review Exercises

5.1) Match the items in the column on the left with the items in the column on the right. Some items match to multiple items.

(a) $\text{Proj}_{\vec{x}}\vec{y}$

(b) A set of vectors includes the zero vector.

(c) $\det A \det B$

(d) Every column of A has a pivot

(e) A basis for $\text{Col}(A)$.

(f) U is an orthogonal matrix.

(g) Orthogonal complement $W^{\perp}$

(h) $(\text{Row}A)^{\perp}$

(i) $(\text{Col}A)^{\perp}$

(j) Orthonormal vectors

(k) A is singular

(l) 0 is **not** an eigenvalue of A

(m) 0 is an eigenvalue of A

(n) PD^kP^{-1}

(o) A is a 3×4 matrix with linearly independent columns.

(p) Orthogonal projection of $\vec{y}$ onto V

(q) A does not have an LU decomposition

(r) A has the decomposition $A = PDP^{-1}$

(s) T is a linear transformation whose standard matrix, A, is one-to-one.

(I) $\{\vec{x} \;:\; \vec{x} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\}$

(II) $\dfrac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}\vec{x}$

(III) Unit length, pairwise orthogonal

(IV) $\det(A) = 0$

(V) Not possible

(VI) $\det(A) \neq 0$

(VII) $\left(PDP^{-1}\right)^k$

(VIII) Row swaps are needed to express A in echelon form.

(IX) $\text{Null}A$

(X) The vector $\widehat{y} \in V$ closest to $\vec{y}$.

(XI) $\text{Null}A^T$

(XII) The eigenvalues of A are distinct.

(XIII) Its columns are orthonormal.

(XIV) $\text{Null}A$ is **not** just the zero vector.

(XV) $\text{Null}A$ is just the zero vector.

(XVI) The vectors are linearly dependent.

(XVII) The system $A\vec{x} = \vec{0}$ has only the trivial solution.

(XVIII) The columns of A are linearly independent.

(XIX) The pivot columns of A.

(XX) $\det(AB)$