# Sample Final B, Math 1554

**PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS**

First Name ____________________ Last Name ____________________

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*You do not need to justify your reasoning for questions on this page.*

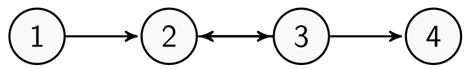
1. (10 points) Determine whether the statements are true or false.

| | true | false |
|---|---|---|
| a) If $A = A^T$, $\vec{x} \neq \vec{y}$, $A\vec{x} = 3\vec{x}$, and $A\vec{y} = -\vec{y}$, then $\vec{x}^T\vec{y} = 0$. | ◯ | ◯ |
| b) If $\vec{x}_1$ maximizes a quadratic form, $Q(\vec{x})$, subject to the constraint $\|\|\vec{x}\|\| = 1$, then so does $-\vec{x}_1$. | ◯ | ◯ |
| c) If $U$ is an echelon form of matrix $A$, then Col$(U)$ = Col$(A)$. | ◯ | ◯ |
| d) $S = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1\}$ is a subspace of $\mathbb{R}^3$. | ◯ | ◯ |
| e) An eigenspace is a subspace spanned by a single eigenvector. | ◯ | ◯ |
| f) Row operations on a $n \times n$ matrix do not change its eigenvalues. | ◯ | ◯ |
| g) If $A \in \mathbb{R}^{n\times n}$ and $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$, then $A\vec{x} \cdot A\vec{y} = \vec{x}^T A^T A \vec{y}$. | ◯ | ◯ |
| h) If the eigenvalues of $3 \times 3$ matrix $A$ are $1, 2, 4$, and $A = PBP^{-1}$, for $3 \times 3$ matrices $P$ and $B$, then the eigenvalues of $B$ are $1, 2, 4$. | ◯ | ◯ |
| i) The orthogonal compliment of the subspace $H = \{\vec{x} \in \mathbb{R}^3 \mid x_1 = x_2 - x_3\}$ is a line. | ◯ | ◯ |
| j) If $A$ is a $n \times n$ matrix, $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$, and $A\vec{x} = A\vec{y}$ for a particular $\vec{x} \neq \vec{y}$, then dim(Row$(A)^\perp$) $\neq 0$. | ◯ | ◯ |

**You do not need to explain your reasoning for questions on this page.**

2. (10 points) If possible, give an example of the following. If it is not possible, write "not possible".

   (a) A Google Matrix, $G$, for the set of web pages that link to each other according to the diagram below.

   (b) A homogeneous linear system with two equations that has no solutions.

   (c) A matrix $A \in \mathbb{R}^{2\times 2}$ that is orthogonally diagonalizable but is not invertible.

   (d) A $2 \times 2$ matrix whose eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 10$, and whose corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

**You do not need to explain your reasoning for questions on this page.**

3. (10 points) If possible, give an example of the following. If it is not possible, write “not possible”.

   (a) A $2 \times 2$ matrix that is stochastic and orthogonal.

   (b) A $5 \times 3$ matrix, $A$, in reduced echelon form, such that dim(Col$(A)^\perp$) $= 1$.

   (c) A quadratic form $Q : \mathbb{R}^3 \mapsto \mathbb{R}$, with no cross-terms, that has maximum value 5, subject to the constraint that $||\vec{x}|| = 1$.

   (d) A matrix that is the standard matrix for the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$. $T$ first rotates points counterclockwise by $\pi$ radians, and then reflects them through the line $x_1 = x_2$.

   (e) A $3 \times 2$ matrix whose column space is spanned by the vector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and whose null space is the line $x_1 = 4x_2$.

**You do not need to explain your reasoning for questions on this page.**

4. Below is a SVD factorization for a matrix $A = U\Sigma V^T$, where

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix}$$

Fill in the blanks.

(a) (1 point) What is the rank of $A$? _______

(b) (1 point) What is the largest value of $\|A\vec{x}\|$, subject to $\|\vec{x}\| = 1$? _______

(c) (1 point) List an orthonormal basis for Row$A$. ___________________

(d) (1 point) List an orthonormal basis for Col$A$. ___________________

5. (2 points) If possible, fill in the missing entries of the $3 \times 3$ matrices with numbers so that the matrices are singular. If it is not possible to do so, write "not possible" below the matrix.

$$B = \begin{bmatrix} 0 & & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad\qquad C = \begin{bmatrix} 3 & 3 & \\ 2 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

6. (4 points) Fill in the blanks. You do not need to explain your reasoning.

(a) Suppose $\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. The area of the parallelogram determined by $\vec{0}, \vec{u}, \vec{v}$, and $\vec{u} + \vec{v}$ is __________.

(b) If $Q$ is $n \times n$ and orthogonal, then $Q^T Q$ equals _______________.

(c) If $A$ is square and singular, at least one eigenvalue of $A$ is _______________.

(d) If $A$ is $3 \times 3$, dim(Col($A$)) = 1, then dim(Row($A^T$)) = _______________.

7. (5 points) If possible, compute the $LU$ factorization of $A$. Clearly indicate matrices $L$ and $U$.

$$A = \begin{bmatrix} 2 & 4 & 9 & 1 \\ 2 & 1 & 0 & 0 \\ 4 & 5 & 9 & 0 \end{bmatrix}$$

8. $H = \{\vec{x} \in \mathbb{R}^3 \ : \ x_1 - x_2 - 9x_3 = 0\}$.

   (a) (3 points) Construct a basis for $H$.

   (b) (2 points) Write down a basis for $H^\perp$.

9. (6 points) Suppose there are two cities, $A$ and $B$. Every year,

   - 60% of the people from $A$ move to $B$, and 40% stay in $A$.
   - 10% of the people from $B$ move to $A$, and 90% stay in $B$.

   The initial populations of $A$ and $B$ are $a_0$ and $b_0$, respectively.

   (a) What is the stochastic matrix for this situation?

   (b) After a long period of time, what is the population in city $A$?

10. (3 points) Identify a non-zero value of $k$ so that $P$ has one real eigenvalue of algebraic multiplicity 2, and another real eigenvalue with algebraic multiplicity 1. Show your work.

$$P = \begin{pmatrix} 3 & 1 & 0 \\ k & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

11. (10 points) Construct an SVD for $A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}$.

12. (10 points) Find the least squares solution, $\hat{x}$, to the equation below. Don't forget to show your work.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{\hat{x} = \begin{pmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{pmatrix}}$$

13. (10 points) Let $A = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$.

*This question is from sample midterm 3A, its solution is on Canvas and Piazza.*

(a) Calculate the eigenvalues of $A$.

(b) If possible, construct matrices $P$ and $C$ such that $A = PCP^{-1}$.

*Sample Final B, Answers*

1. True/false.
   (a) True
   (b) True
   (c) False
   (d) False
   (e) False
   (f) False
   (g) True
   (h) True
   (i) True
   (j) True
2. Example Construction.
   (a) $G = \frac{0.85}{4}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 4 & 0 & 2 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} + \frac{0.15}{4}K$, where $K$ is a $4 \times 4$ matrix of ones.
   (b) Not possible (every homogeneous system has a trivial solution)
   (c) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
   (d) $A = PDP^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & \\ & 10 \end{pmatrix}\begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$
3. Example Construction.
   (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
   (b) Not possible
   (c) $Q = 5x_1^2$
   (d) $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
   (e) $\begin{pmatrix} 2 & -8 \\ 0 & 0 \\ 1 & -4 \end{pmatrix}$
4. Fill in the blanks:
   (a) 3

(b) 4

(c) $v_1, v_2, v_3$

(d) $u_1, u_2, u_3$

5. Matrix completion.

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 3 & 3 \\ 2 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

6. Fill in the blanks.

(a) 2

(b) $I_n$

(c) $0$

(d) $1$ since it has one pivot.

7. LU factorization.

$$\begin{aligned} A &= \begin{pmatrix} 2 & 4 & 9 & 1 \\ 2 & 1 & 0 & 0 \\ 4 & 5 & 9 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 2 & 4 & 9 & 1 \\ 0 & -3 & -9 & -1 \\ 0 & -3 & -9 & -2 \end{pmatrix} \\ &\sim \begin{pmatrix} 2 & 4 & 9 & 1 \\ 0 & -3 & -9 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= U \end{aligned}$$

By inspection, based on the row operations we used to produce $U$ from $A$,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

8. (a) One equation in three variables: we seek two basis vectors. Choose $x_2 = 1$ and $x_3 = 0$ to obtain $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Choose $x_2 = 0$ and $x_3 = 1$ to obtain

$$v_2 = \begin{pmatrix} 9 \\ 0 \\ 1 \end{pmatrix}$$

A basis for $H$ is $\{v_1, v_2\}$.

(b) The basis for $H^\perp$ has only one non-zero vector, $v_3$. Any non-zero vector that is perpendicular to both $v_1$ and $v_2$ will suffice. By inspection,

$$v_3 = \begin{pmatrix} 1 \\ -1 \\ -9 \end{pmatrix}$$

9. Markov Chain.

(a) $P = \frac{1}{10}\begin{pmatrix} 4 & 1 \\ 6 & 9 \end{pmatrix}$

(b) Solve

$$(P - I)\vec{x} = \vec{0}$$
$$0.10\begin{pmatrix} -6 & 1 \\ 6 & -1 \end{pmatrix}\vec{x} = \vec{0}$$

Thus, $\vec{x} = \frac{1}{7}\begin{pmatrix} 1 \\ 6 \end{pmatrix}$. After a long period of time, the population in city $A$ is

$$\frac{a_0 + b_0}{7}$$

10. Set up the usual determinant:

$$\begin{aligned} 0 &= |P - \lambda I| \\ &= (\lambda - 5)((3-\lambda)(3-\lambda) - k) \\ &= (\lambda - 5)(9 - 6\lambda + \lambda^2 - k) \end{aligned}$$

If $k = 4$, then $\lambda = 5$ has algebraic multiplicity two.

11. SVD:

$$A^T A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$$

By solving characteristic equation, $\lambda = 4, 9$.

$$\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$V$ matrix:

$$A^T A - 9I = \begin{pmatrix} -4 & 2 \\ * & * \end{pmatrix}, \quad * = \text{not needed}$$

Thus, an eigenvector is $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. And $A^T A$ is symmetric, so $\vec{v}_1 \cdot \vec{v}_2 = 0$. So $\vec{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Scale these vectors so they have unit length, place them into a matrix,

$$V = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$U$ matrix:

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{3\sqrt{5}}\begin{pmatrix}1 & 2\\ 2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} = \frac{1}{3\sqrt{5}}\begin{pmatrix}5\\ 2\\ 4\end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{2\sqrt{5}}\begin{pmatrix}1 & 2\\ 2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}2\\ -1\end{pmatrix} = \frac{1}{2\sqrt{5}}\begin{pmatrix}0\\ 4\\ -2\end{pmatrix}$$

$\vec{u}_3 \in \text{Col}A^{\perp}$, so set

$$\vec{u}_3 = \begin{pmatrix}a\\ b\\ c\end{pmatrix}$$

and solve equations

$$\vec{u}_1 \cdot \vec{u}_3 = \vec{u}_2 \cdot \vec{u}_3 = 0$$

This yields

$$\begin{aligned}0 &= 5a + 2b + 4c\\ 0 &= 4b - 2c\end{aligned}$$

One variable is free: set $c = 2$, then $b = 1$ and $a = 2$.

$$\vec{u}_3 = \frac{1}{3}\begin{pmatrix}-2\\ 1\\ 2\end{pmatrix}$$

The $U$ matrix is

$$U = \begin{pmatrix}\frac{5}{3\sqrt{5}} & \frac{0}{2\sqrt{5}} & \frac{-2}{3}\\ \frac{2}{3\sqrt{5}} & \frac{4}{2\sqrt{5}} & \frac{1}{3}\\ \frac{4}{3\sqrt{5}} & \frac{-2}{2\sqrt{5}} & \frac{2}{3}\end{pmatrix}$$

The SVD is

$$A = U\Sigma V^T$$

with $U$, $\Sigma$, and $V$ as above. There is no need to re-write each element out again, but students should indicate that $A = U\Sigma V^T$ somewhere for full points.

12. $A^TA = \begin{pmatrix}3 & 2 & 0\\ 2 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}$, $A^Tb = \begin{pmatrix}5\\ 5\\ 4\end{pmatrix}$. Solving the normal equation we find $\hat{x} = \begin{pmatrix}1\\ 1\\ 2\end{pmatrix}$

13. $0 = |A - \lambda I| = (1-\lambda)(3-\lambda) + 10$
    $0 = \lambda^2 - 4\lambda + 13$
    $\lambda = 2 \pm \frac{1}{2}\sqrt{16-52} = 2 \pm \frac{1}{2}\sqrt{-36} = 2 \pm 3i$
    Solve $(A - \lambda I)\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$ to find eigenvector.

Choose $\lambda = 2 - 3i$ (you can choose $\lambda = 2 + 3i$ if you prefer)

$$A - \lambda I = \begin{pmatrix} -1 + 3i & 5 \\ -2 & 1 + 3i \end{pmatrix}$$

Thus, using 2nd row, $-2x_1 + (1 + 3i)x_2 = 0$. We only need one of the two rows because the two are multiples of each other. A possible solution is $\vec{v} = \begin{pmatrix} 1 + 3i \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$$\implies P = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}, C = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$